

Universal Laws of Thermodynamics

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Thermodynamics is one of the most successful physical theories ever formulated. Though it was initially developed to deal with steam engines and, in particular, the problem of conversion of heat into mechanical work, it has prevailed even after the scientific revolutions of relativity and quantum mechanics. Despite its wide range of applicability, it is known that the laws of thermodynamics break down when systems are correlated with their environments. In the presence of correlations, anomalous heat flows from cold to hot baths become possible, as well as memory erasure accompanied by work extraction instead of heat dissipation.

Here, we generalize thermodynamics to physical scenarios which allow presence of correlations, including those where strong correlations are present. We exploit the connection between information and physics, and introduce a consistent redefinition of heat dissipation by systematically accounting for the *information flow* from system to bath in terms of the conditional entropy. As a consequence, the formula for the Helmholtz free energy is accordingly modified. Such a remedy not only fixes the apparent violations of Landauer's erasure principle and the second law due to anomalous heat flows, but it also leads to a reformulation of the laws of thermodynamics that are *universally* respected. In this information-theoretic approach, correlations between system and environment store work potential. Thus, in this view, the apparent anomalous heat flows are the refrigeration processes driven by such potentials.

I. INTRODUCTION

Thermodynamics lays one of the basic foundations of our current understanding of the physical world. It prevails despite the revolutions that took place in science through relativity and quantum mechanics. Beyond its own domain, so far, it has been successfully applied to black holes [1, 2], tiny quantum engines comprised of only a few qubits [3–6]. It was initially developed to deal with macroscopic heat engines, in particular, to investigate conversion of energy into mechanical work and heat, long before quantum mechanics had been developed. It is therefore plausible that thermodynamics in the microscopic regime, where the quantum properties dominate, departs significantly from its macroscopic counterpart. This is indeed the case. Inspired by resource theories [7–13], recently developed in the domain of quantum information [14–16], a renewed effort has been made to understand the foundations of thermodynamics in the quantum domain, giving rise to many interesting results [17–27], including its connections to statistical mechanics [28–31] and information theory [32–42]. Studying extractable work from a thermal machine involving individual quantum system not only requires modified forms of free energies [18, 19], but also relies on many second laws to dictate state transformations [22, 24, 25]. Notably, all the results converge to their macroscopic analogues for large ensemble of identical systems [17]. Heat, in the microscopic regime, was initially studied in connection with a system's information by Landauer [39] to validate the second law by exorcising Maxwell's demon [34, 37, 38], which led to *Landauer's erasure principle*. It was then further extended to information theory in the context of information erasure [32–36, 41]. Heat dissipation due to erasing information from

individual quantum systems has recently been studied systematically [32].

An important feature in the microscopic regime is that the quantum particles can exhibit non-trivial correlations, such as entanglement [9] and other quantum correlations [43]. Thermodynamics in the presence of correlations has been considered only in limited physical situations. It is assumed, in nearly all cases of thermodynamical processes, that system and bath are initially uncorrelated, although correlations may appear in the course of the process. In fact, it has been noted that in the presence of such correlations, Landauer's erasure principle could be violated [32]. Even more strikingly, with strong quantum correlation between two thermal baths of different temperatures, heat could flow from the colder bath to the hotter one [44–47]. In other words, the fundamental laws of thermodynamics are violated in the presence of system-bath correlations. There is an effort to resolve this issue using the concept of relative thermalization [48], but it is far from being a complete remedy to the violations of the fundamental laws.

The aim of the present work is to re-establish the laws of thermodynamics in all possible physical scenarios, including initial inter-system correlations. Our first step employs a consistent redefinition of heat dissipation based on a systematic account of *information flow*. Thereby we establish a concrete inter-relation between thermodynamics and information. This information-theoretic approach towards thermodynamics also results in a redefinition of the Helmholtz free energy, and therefore a consistent quantification of work. Our approach not only fixes the apparent violations of Landauer's erasure principle and the second law due to anomalous heat flows, but also leads to reformulations of the laws of thermodynamics that are *universally* respected.

The remainder of the article is organized as follows. We start with briefly reviewing the laws of thermodynamics in

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section II, and defining the notion of heat in section III, where we also justify why it should be the correct notion compared to the others considered in the literature. Then in section IV, we sketch the violations of the laws of thermodynamics and Landauer’s erasure principle in the presence of system-bath correlations. To resolve these contradictions, we propose *universal* laws of thermodynamic, under which we can restore Landauer’s erasure principle and the laws of thermodynamics by systematically accounting for the information stored in the correlations, in section V. We conclude in section VI.

II. LAWS OF THERMODYNAMICS

The theory of thermodynamics can be summarized in its three main laws. The *zeroth law* introduces the notion of thermal *equilibrium* as an equivalence relation of states, where *temperature* is the parameter that labels the different equivalence classes. In particular, the transitive property of the equivalence relation implies that if a body A is in equilibrium with a body B , and B is with a third body C , then A and C are also in equilibrium. The *first law* assures energy conservation. It states that in a thermodynamic process not all of energy changes are of the same nature and distinguishes between *work*, the type of energy that allows for “useful” operations as raising a weight, and its complement *heat*, any energy change which is not work. Finally, the *second law* establishes an arrow of time. It has several formulations and perhaps the most common one is the *Clausius* statement, which reads: *No process is possible whose sole result is the transfer of heat from a cooler to a hotter body*. Such a restriction not only introduces the fundamental limit on how and to what extent various forms of energy can be converted to accessible mechanical work, but also implies the existence of an additional state function, the *entropy*, which has to increase. There is also *third law* of thermodynamics; we shall, however, leave it out of the discussion, as it is beyond immediate context of the physical scenarios considered here.

Although the laws of thermodynamics were developed phenomenologically, they have profound implications in information theory. The paradigmatic example is the Landauer erasure principle, which states: “Any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information-bearing degrees of freedom of the information-processing apparatus or its environment” [34]. Therefore, an erasing operation is bound to be associated with a heat flow to the environment.

III. HEAT

In thermodynamics, *heat* is defined as the flow of energy from a system to its environment, normally considered as a thermal *bath* at certain temperature, in some way different from *work*. Work, on the other hand, is quantified as *the flow of energy, say to a bath or to an external agent, that could be extractable (or accessible)*. Operationally, it is en-

ergy that can be used to raise a weight or similar battery system. Therefore, the difference between change in internal energy and extractable work quantifies the heat flow to the bath. More explicitly, let us consider a thermal bath with Hamiltonian H_B and at temperature T represented by the Gibbs state $\rho_B = \tau_B = \frac{1}{Z_B} \exp(\frac{-H_B}{kT})$, where k the Boltzmann constant and $Z_B = \text{Tr}[\exp(\frac{-H_B}{kT})]$ is the partition function. Then for a process that transforms the thermal bath $\rho_B \rightarrow \rho'_B$ with the fixed Hamiltonian H_B , the heat transfer to the bath is quantified (see section A) as

$$\Delta Q = kT \Delta S_B, \quad (1)$$

where $\Delta S_B = S(\rho'_B) - S(\rho_B)$ is the change in bath’s von Neumann entropy, $S(\rho_B) = -\text{Tr}[\rho_B \log_2 \rho_B]$. The work stored in the bath is ΔF_B , where $F(\rho_B) = E(\rho_B) - kT S(\rho_B)$ is the Helmholtz free energy, with $E(\rho_B) = \text{Tr}(H_B \rho_B)$. Heat expressed in Eq. (1) is the correct quantification of heat, which can be justified in two ways. First, it has a clear information-theoretic interpretation, which accounts for the information flow to the bath. Second, it is the flow of energy to the bath other than work and, with the condition of entropy preservation, any other form of energy flow to the bath will be stored as extractable work, and thus not converted into heat (see also section V C). Note that there is another definition of heat that can be found in the literature, e.g. [32, 46], where heat is defined as the change in the internal energy of the bath. In appendix A, we extensively compare the two definitions; both coincide in the scenario of a very large bath that always remains in equilibrium.

IV. VIOLATIONS OF LAWS OF THERMODYNAMICS

In order to highlight how the laws of thermodynamics break down in the presence of correlations, let us discuss the following two examples. In the first, the system S is purely classically correlated with the bath B at temperature T , while in the other they are jointly in a pure state and share quantum entanglement. In both the examples the Hamiltonians of the system and bath (H_S and H_B) remain unchanged throughout the processes.

Example 1 – Classical correlations.

$$\rho_{SB} = \sum_i p_i |i\rangle\langle i|_S \otimes |i\rangle\langle i|_B \xrightarrow{U_{SB}^c} \rho'_{SB} = |\phi\rangle\langle\phi|_S \otimes \sum_i p_i |i\rangle\langle i|_B,$$

Example 2 – Entanglement.

$$|\Psi\rangle_{SB} = \sum_i \sqrt{p_i} |i\rangle_S |i\rangle_B \xrightarrow{U_{SB}^e} |\Psi\rangle'_{SB} = |\phi\rangle_S \otimes |\phi\rangle_B,$$

where in both examples $|\phi\rangle_X = \sum_i \sqrt{p_i} |i\rangle_X$ with $X \in \{S, B\}$ and $1 > p_i \geq 0$ for all i . Note that the unitaries, U_{SB}^c and U_{SB}^e , leave the local energies of system and bath unchanged, and U_{SB}^c does not change the bath state.

1. Violations of first law

In *Example 1*, the Helmholtz free energy of the system increases $F(|\phi\rangle_S) > F(\rho_S)$ and therefore a work $-\Delta W_S = \Delta F_S > 0$ is performed on the system. To assure the energy conservation of the system, an equal amount of heat is required to be transferred to the bath. Surprisingly, however, no heat is transferred to the bath as it remains unchanged. Thus $\Delta E_S \neq -\Delta W_S - \Delta Q$, i.e. the energy conservation is violated and so the first law.

A further violation can also be seen in *Example 2* involving system-bath quantum entanglement. In this case, a non-zero work $-\Delta W_S = \Delta F_S > 0$ has been performed on the system, and a heat flow to the bath is expected. In contrast, there is a *negative* heat flow to the bath! Therefore, it violates the first law, i.e. $\Delta E_S \neq -\Delta W_S - \Delta Q$.

2. Violations of second law and anomalous heat flows

We now show how correlation could result in a violation of the *Kelvin-Planck* statement of the second law, which states: *No process is possible whose sole result is the absorption of heat from a reservoir and the conversion of this heat into work*. In *Example 1*, no change in the local bath state indicates that there is no transfer of heat. However, the change in the Helmholtz free energy of the local system is $-\Delta W_S = \Delta F_S > 0$. Thus, a non-zero amount of work is performed on the system without even absorbing heat from the bath ($\Delta Q = 0$).

The situation becomes more striking in *Example 2*, with initial system-bath entanglement. In this case, $-\Delta W_S = \Delta F_S > 0$ amount of work is performed on the system. However, not only is there no heat flow from the bath to the system, but there is a *negative* heat flow to the bath! Thus, the second law is violated.

We next see how the presence of correlations can lead to anomalous heat flows and thereby a violation of the *Clausius* statement based second law (see Fig. 1). Such violations are known for the other definition of heat $\Delta Q = \Delta E_B$ (see [46] and references therein). Here we show that such violations are also there with new heat definition $\Delta Q = kT\Delta S_B$. Let $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ be an initial bipartite finite dimensional state whose marginals $\rho_A = \text{Tr}_B \rho_{AB} = \frac{1}{Z_A} \exp[-\frac{H_A}{kT_A}]$ and $\rho_B = \frac{1}{Z_B} \exp[-\frac{H_B}{kT_B}]$ are thermal states at different temperatures T_A and T_B and with Hamiltonians H_A and H_B . In absence of initial correlations between the baths A and B , any energy preserving unitary will respect Clausius' statement of the second law. However, if initial correlations are present, this will not be necessarily the case.

Consider a state transformation $\rho'_{S_B} = U_{AB} \rho_{AB} U_{AB}^\dagger$ where U_{AB} is a energy preserving unitary acting on ρ_{AB} . As the thermal state minimizes the free energy, the final reduced states ρ'_S and ρ'_B have increased their free energy,

$$\Delta E_A - kT_A \Delta S_A \geq 0 \quad (2)$$

$$\Delta E_B - kT_B \Delta S_B \geq 0, \quad (3)$$

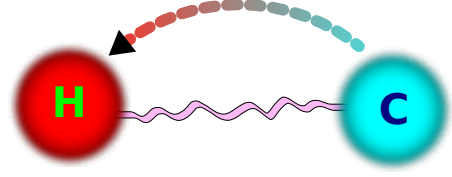


Figure 1. In presence of correlations, spontaneous heat flows from cold to hot baths are possible [46]. This is a clear violation of second law.

where $T_{A/B}$ is the initial temperature of the baths, and $\Delta E_{A/B}$ and $\Delta S_{A/B}$ are the change in internal energy and entropy respectively.

By adding Eqs. (2) and (3), and considering energy conservation, we get

$$T_A \Delta S_A + T_B \Delta S_B \leq 0. \quad (4)$$

Due to the conservation of total entropy, the change in mutual information is simply $\Delta I(A : B) = \Delta S_A + \Delta S_B$, with $I(A : B) = S_A + S_B - S_{AB}$. This allows us to rewrite Eq. (4) in terms of only the entropy change in A as

$$(T_A - T_B) \Delta S_A \leq -T_B \Delta I(A : B). \quad (5)$$

If the initial state $\rho_{AB} = \rho_A \otimes \rho_B$ is uncorrelated, then the change in mutual information is necessarily positive $\Delta I(A : B) \geq 0$, and

$$k(T_A - T_B) \Delta S_A = \Delta Q_A \frac{T_A - T_B}{T_A} \leq 0. \quad (6)$$

To see that this equation is precisely the Clausius statement, consider without loss of generality that A is the hot bath and $T_A - T_B > 0$. Then, inequality (6) implies an entropy reduction of the hot bath $\Delta S_A \leq 0$ i. e. a heat flow from the hot bath to the cold one.

However, if the the system is initially correlated, the process can reduce the mutual information, $\Delta I(A : B) < 0$, and Eq. (5) allows a heat flow from the cold bath to the hot one.

3. Violations of zeroth law

The zeroth law establishes the notion of thermal equilibrium as an equivalence relation, in which temperature labels the different equivalent classes. To see that the presence of correlations also invalidates the zeroth law, we show that the transitive property of the equivalence relation is not fulfilled. Consider a bipartite system AC in an initial correlated state ρ_{AC} , like in *Examples 1* and *2*, and a third party B which is in a thermal state at the same temperature of the marginals ρ_A and ρ_C (see Fig. 2). Then, while the subsystems AB and BC are mutually in equilibrium, the subsystems AC are not, clearly violating transitivity. There are several ways to realize that the parties AC are not in equilibrium. One way is to see that any energy preserving unitary, except for the identity, decreases the amount of correlations between the parties,

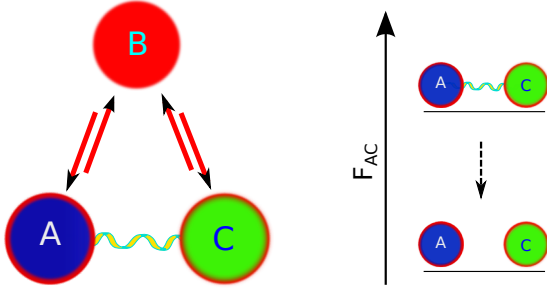


Figure 2. In presence of correlations, the notion of equilibrium is not an equivalence relation. While A is in equilibrium with B and B is in equilibrium with C , A and C are not in equilibrium. The transitive property is violated. This is justified, on the right, as $F(\rho_{AC}) > F(\rho_A \otimes \rho_C)$.

$\Delta I(A : C) < 0$, which implies that the initial state is not stable. This can be shown from Eq. (4) for the particular case of equal temperatures and the definition of mutual information. Another way is to see that the Helmholtz free energy follows $F(\rho_{AC}) > F(\rho_A \otimes \rho_C)$.

4. Violations of Landauer's erasure principle

Another thermodynamic principle that breaks down when correlations are present is Landauer's erasure principle. Landauer postulated that in order to erase one bit of information in presence of a bath at temperature T , an amount of heat needed to be dissipated is $kT \log 2$. As noted in [32], when the system is classically correlated, there exists erasing process which does not increase entropy of the bath (see *Example 1*). The situation becomes more striking when the system shares quantum entanglement with the bath. This is the case of *Example 2* with initial entanglement, where instead of increasing, an erasing process reduces the entropy of the bath and the bath is cooled down.

V. UNIVERSAL QUANTUM THERMODYNAMICS

To summarize the previous discussion, the violations of the laws of thermodynamics indicate that correlations between two systems, irrespective of the corresponding marginals being thermal states or not, manifest out-of-equilibrium phenomena. In order to re-establish the laws of thermodynamics, one not only has to look at the local marginal systems, but also the correlations between them. To do so, we start with re-defining heat by properly accounting for the information flow and thereby restoring Landauer's erasure principle. Then we proceed to restore the fundamental laws.

The transformations considered in the rest of the paper are *entropy preserving* operations. More explicitly, given a system-bath setting in a state ρ_{SB} , we consider transformations $\rho'_{SB} = \Lambda(\rho_{SB})$ such that the von Neumann entropy is unchanged i. e. $S(\rho'_{SB}) = S(\rho_{SB})$. The Hamiltonians of the system and the bath are the same before and after the trans-

formation $\Lambda(\cdot)$. Note that we do not demand energy conservation, rather assuming that a suitable battery takes care of that. In fact, the work cost of such an operation $\Lambda(\cdot)$ is quantified by the global internal energy change $\Delta W = \Delta E_S + \Delta E_B$. Another comment to make is that we implicitly assume a bath of unbounded size; namely, it consists of the part ρ_B of which we explicitly track the correlations with S , but also of arbitrarily many independent degrees of freedom.

An important remark is that this set of entropy preserving operations is strictly larger than the unitary transformations considered in many texts, and in particular in the so-called thermal operations [17]. Although, both coincide in the asymptotic limit of many copies or runs. In this work we are implicitly considering always the asymptotic scenario of $n \rightarrow \infty$ copies of the state in question ("thermodynamic limit").

A. Universal Landauer's erasure principle and heat

In extending thermodynamics in correlated scenarios and linking thermodynamics with information, we consider the *quantum conditional entropy* as the natural quantity to represent information content in the system. For a joint system-bath state ρ_{SB} , the information content in the system S , given all the information available in the bath B at temperature T , is quantified by the conditional entropy $S(S|B) = S(\rho_{SB}) - S(\rho_B)$. Note that this quantity not only counts the information content in the local system, but also takes into account the information stored in the correlation. It vanishes when the joint system-environment state is perfectly classically correlated. However, it can even become negative in the presence of entanglement. With quantum conditional entropy as the correct quantifier of system information, the *second law* of information can be restored as follows.

Lemma 1 (Universal informational second law). *Let $\rho_{SB} \in \mathcal{H}_S \otimes \mathcal{H}_B$ be an initial bipartite system-bath state and Λ^{SB} an entropy preserving operation acting on ρ_{SB} , and resulting in $\rho'_{SB} = \Lambda^{SB}(\rho_{SB})$. Then, with the reduced states before (after) the evolution denoted ρ_S (ρ'_S) and ρ_B (ρ'_B), respectively, we have*

$$\Delta S_B = -\Delta S(S|B), \quad (7)$$

where $\Delta S_B = S(\rho'_B) - S(\rho_B)$ is the change in (von Neumann) entropy of the bath, and $\Delta S(S|B) = S(S'|B') - S(S|B)$ is the change in conditional entropy of the system.

An increase (decrease) in bath entropy is compensated by a decrease (increase) in conditional entropy of the system. Therefore information content is preserved under entropy preserving (EP) operations. Note that in the presence of initial correlations, the informational second law could be violated if one considers only system entropy (see appendix B). Thus, a *universal* Landauer's principle is required to be expressed in terms of conditional entropy of the system, rather than its local entropy, and is stated in Theorem 2.

Theorem 2 (Universal Landauer’s erasure principle). *The dissipated heat associated to information erasure of a system S connected to a bath B at temperature T by an entropy preserving operation, is equal to*

$$\Delta Q = kT S(S|B), \quad (8)$$

where $\Delta S(S|B)$ is the change in conditional entropy of the system.

Proof. It follows immediately from the definition of heat in Eq. (1), and Lemma 1, noting that the final conditional entropy vanishes, $S(S'|B') = 0$. \square

Now one can easily see that Landauer’s erasure principle is respected for arbitrary erasing process, cf. *Examples 1* and *2*.

B. Work extraction

Next, we address extraction of work from a system S coupled to a bath B at temperature T . The system may share correlations, be they classical or quantum, with the bath. Without loss of generality, we assume that the system Hamiltonian H_S is unchanged in the process. Note that the extractable work has two contributions: one comes from system-bath correlations and the other from the local system alone, irrespective of its correlations with the bath. Here we consider these two contribution separately.

Here we put forward the following lemma which will be useful for our work extraction protocols later.

Lemma 3. *For an arbitrary system-bath state ρ_{SB} , there exists an additional systems A with degenerate Hamiltonian $H_A = 0$, and a state ρ_A , such that there is an energy- and entropy-preserving transformation $\rho_A \otimes \rho_{SB} \xrightarrow{\Lambda^{ASB}} \rho'_A \otimes \rho_S \otimes \rho_B$ with*

$$S(AS|B)_{\rho_A \otimes \rho_{SB}} = S(A'S|B)_{\rho'_A \otimes \rho_S \otimes \rho_B}. \quad (9)$$

Proof. It is evident that if A is large enough and we start it in a thermal state, the condition for entropy preservation can be satisfied.

As the local system and bath states remain unchanged under the transformation, and the system A has trivial Hamiltonian, also the (expected) energy is preserved. \square

The change in entropy of the additional system is exactly equal, and opposite in sign, to the change in system-bath correlation (which is nothing but the mutual information): $S_A - S_{A'} = S_S - S(S|B) = I(S : B)$.

1. Extractable work from correlation

The extractable work only from the correlation (see Fig. 3) is given by the following result. Here, by extracting work from the correlation, we mean any process that returns the system and the bath in the original reduced states, ρ_S and $\rho_B = \tau_B$, respectively.

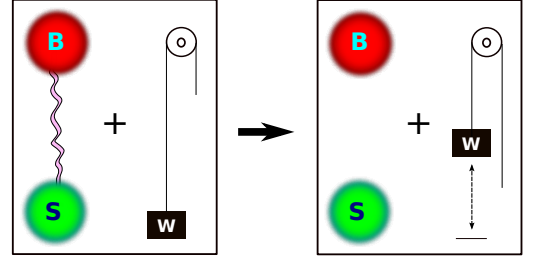


Figure 3. Correlations can be understood as a work potential. See section VB for details.

Theorem 4. *For a system-bath state ρ_{SB} , with the bath at temperature T , the maximum extractable work solely from the correlation, using entropy preserving operations, is given by*

$$\Delta W_C = kT I(S : B), \quad (10)$$

where $I(S : B) = S_S + S_B - S_{SB}$ is the mutual information.

Proof. Consider the following work extraction protocol in three steps:

Step 1. We attach to ρ_{SB} an ancillary system A with trivial Hamiltonian $H_A = 0$, consisting of $I(B : S)$ qubits in the maximally mixed state $\tau_A = \left(\frac{1}{2}\right)^{\otimes I(S:B)}$ (which is thermal!).

Step 2. By using a global entropy preserving operation, according to Lemma 3, we convert $\tau_A \otimes \rho_{SB}$ into $|\phi\rangle\langle\phi|_A \otimes \rho_S \otimes \rho_B$, i.e. turning the additional state into a pure state $\rho'_A = |\phi\rangle\langle\phi|$ of A while leaving the marginal system and bath states unchanged. Clearly, the extractable work stored in the correlation is now transferred to the new additional system state ρ'_A .

Step 3. Work is extracted from ρ'_A at temperature T , equal to $\Delta W_C = I(S : B)_{\rho_{SB}} kT$. \square

2. Universal Helmholtz free energy

The maximum extractable work from a state ρ_S , disregarding the correlations with a bath at temperature T , is given by $\Delta W_L = F(\rho_S) - F(\tau_S)$, where $\tau_S = \frac{1}{Z_S} \exp[-\frac{H_S}{kT}]$ is the corresponding thermal state of the system in equilibrium with the bath. Now, in addition to this “local work”, we have the work due to correlations, and so the total extractable work $\Delta W_S = \Delta W_C + \Delta W_L$, comprising both the system and the correlation, becomes

$$\Delta W_S = \Delta W_L + kT I(S : B)_{\rho_{SB}}. \quad (11)$$

Note that, for the system alone, the Helmholtz free energy $F(\rho_S) = E_S - kT S_S$. However, in the presence of correlations, the free energy is modified to

$$\mathcal{F}(\rho_S) = E_S - kT S(S|B). \quad (12)$$

The above is simply achieved by adding $kT I(S : B)_{\rho_{SB}}$ to $F(\rho_S)$. Therefore, for a system-bath state ρ_{SB} , maximum extractable work from the system can be given as $\Delta W_S =$

$\mathcal{F}(\rho_S) - \mathcal{F}(\tau_S)$, where $\mathcal{F}(\tau_S) = F(\tau_S)$. Then for a transformation, where initial and final states are ρ_{SB} and σ_{SB} respectively, the maximum extractable work from the system, is $\Delta W_S = -\Delta \mathcal{F} = \mathcal{F}(\rho_S) - \mathcal{F}(\sigma_S)$.

We observe that all this is of course consistent with what we know from situations with an uncorrelated bath. Indeed, we can simply make the conceptual step of calling SB “the system”, allowing for arbitrary correlations between S and B , with a suitable infinite bath B' that is uncorrelated from SB . Then, the free energy as we know it is

$$\begin{aligned} F(\rho_{SB}) &= E_S + E_B - kT \mathcal{S}(\rho_{SB}) \\ &= E_S - kT \mathcal{S}(S|B) + E_B - kT \mathcal{S}(\tau_B), \end{aligned} \quad (13)$$

where the first term is the modified free energy in Eq. (12), and the second term is the free energy of the bath in its thermal state. As the latter cannot become smaller in any entropy-preserving operation, the maximum extractable work is $-\Delta \mathcal{F}$.

C. Universal first law

As we have shown, restoring the (information based) second law requires proper accounting of information to correctly quantify the heat flow. The lessons learned from the universal Landauer erasure principle, now lead to the following universal *first law* of thermodynamics for closed systems.

Theorem 5 (Universal first law). *Given an entropy preserving operation $\rho_{SB} \rightarrow \rho'_{SB}$, where the local system state follow the transformation $\rho_S \rightarrow \rho'_S$, the distribution of the change in the system’s internal energy into work and heat satisfies*

$$\Delta E_S = -(\Delta W_S - \Delta W_B) - (\Delta W_B + \Delta Q), \quad (14)$$

where the heat dissipated to the bath is given by

$$\Delta Q = -kT \Delta \mathcal{S}(S|B), \quad (15)$$

and the maximum extractable work from the system is

$$\Delta W_S = -(\Delta E_S - kT \Delta \mathcal{S}(S|B)), \quad (16)$$

and the work performed on the bath is $\Delta W_B = \Delta E_B - kT \Delta \mathcal{S}_B$.

Proof. The proof relies on adding Eqs. (15) and (16). The quantity $\Delta W_S = -(\Delta E_S - kT \Delta \mathcal{S}(S|B))$ was shown to be the maximum extractable work in the preceding Section VB. \square

The maximum work ΔW_S is extracted by thermodynamically reversible processes. Any irreversible process leads to $\Delta W_B > 0$, which means that some work has been performed on the bath. Finally, in an equilibration process, which will happen due to spontaneous relaxation of the bath, such amount of work will be transformed into heat and hence cannot be accessed any more. Note that such an equilibration process is not entropy preserving [28].

In the light of the above universal first law, we reconsider the examples from before. In *Example 1*, it indicates that an initial classically correlated state ρ_{SB} is thermodynamically

equivalent to the final product state ρ'_{SB} . It is also clear from the fact that, in both the states, local thermal states, local average energy and moreover the system’s conditional information are same. The case in *Example 2*, which considers an initial entangled state that is unitarily transformed to a final pure product state, is more intriguing. It says that quantum entanglement can be exploited to perform work (due to negative heat flow) on the bath by leaving the system in pure state. The resulting pure state can further be used to extract work. Therefore a system having entanglement with the bath has more work potential than the uncorrelated one with the same marginals.

D. Universal second law

Now taking into account the information content in the marginal system as well as in the correlation with the bath, in terms of conditional entropy of the system, the second law is also modified, as follows:

Theorem 6 (Kelvin-Planck statement of the universal second law). *No process is possible whose sole result is the absorption of heat from a reservoir and the conversion of this heat into work, where heat and work are defined as in Theorem 5.*

Clearly the previously mentioned second law violations, based on *Examples 1* and *2*, are fixed. In *Example 1*, there is no work is done on the system as both initial and final system-bath states are thermodynamically equivalent and the system contains same work potentials. In *Example 2*, one part of the work stored in the correlation is used to refrigerate the bath, while other part is transferred to the system by lowering its entropy.

Theorem 7 (Clausius statement of the universal second law). *No process is possible whose sole result is the transfer of heat from a cooler to a hotter body, where the work potential stored in the correlations, as defined in Theorem 4, does not decrease.*

Proof. Equation (5) in the derivation of the anomalous heat flow of the previous section together with the work potential defined in Theorem 4 lead to

$$\Delta Q_A \frac{T_A - T_B}{T_A} \leq -\Delta W_C, \quad (17)$$

which implies the statement of Theorem 7. \square

Then, we can interpret the anomalous heat flow as a refrigerator driven by the work potential stored in correlations. From the refrigerator perspective, it is interesting to determine its efficiency, that from Eq. (17) leads to

$$\left| \frac{\Delta W_C}{\Delta Q_A} \right| \leq \frac{T_A - T_B}{T_A} \quad (18)$$

which is nothing else than the *Carnot efficiency*. This is a nice reconciliation with traditional thermodynamics. The Carnot efficiency is a consequence of the fact that reversible processes

are optimal, otherwise the perpetual mobile could be build by concatenating a "better" process and a reversed reversible one. Hence, it is natural that the refrigeration process driven by the work stored in the correlations preserves Carnot statement of second law.

E. Universal zeroth law

Our reconstruction of the zeroth law is a redefinition of equilibrium which, as we have previously seen, cannot be an equivalence relation when correlations between systems are present. Therefore the universally respected version of the law becomes:

Definition 8 (Universal zeroth law). *A collection $\{\rho_X\}_X$ of states is said to be in mutual thermal equilibrium with each other, at a certain temperature T , if and only if no work can be extracted from any of their combinations under entropy preserving operations.*

Note that the notion of thermal equilibrium is extended beyond the transitivity relation used in the traditional zeroth law. The thermal equilibria $A \rightleftharpoons B$ and $B \rightleftharpoons C$ not necessarily assure $A \rightleftharpoons C$. Rather, the systems $\{A, B, C\}$ are guaranteed to be in mutual equilibria if and only if no work is can be extracted from individual systems as well as from their arbitrary collections, using entropy preserving operations.

VI. CONCLUSIONS

We have seen that the laws of thermodynamics break down in presence of correlations e. g., anomalous heat flows from cold to a hot baths become possible [46], as well as violations of Landauer's principle [32] or, even, erasing memory leads to work extraction instead of heat dissipation [42]. In this work we have generalized thermodynamics to the physical scenarios that involve correlations, including those where strong correlations are present. To do so, we have exploited the connection between information and physics, and introduced a consistent redefinition of heat dissipation by systematically accounting the *information flow* from system to bath in terms of the conditional entropy. As a consequence, the Helmholtz free energy has been accordingly modified. Such a remedy, not only fixes the violations of Landauer's erasure principle [39] and the second law due to anomalous heat flows, but also leads to the reformulation of the laws of thermodynamics that are *universally* respected. In this information-theoretic approach, correlations between system and environment store work potential and apparent anomalous heat flows are a refrigeration process driven by such a potentials.

An important remark is that all our derivations, so far, have been made in the asymptotic limit of many copies. A relevant question is how the laws of thermodynamics can be formulated for a single system. In our forthcoming paper [49], we will address these questions by discussing consistent notions of one-shot heat, one-shot Landauer erasure, and of one-shot work extraction from correlations.

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Appendix A: Definitions of heat

We have mathematically defined heat in Eq. (1) according to the common description: "flow of energy to a bath some way other than through work". Note, however, that this is not the most extended definition of heat that one finds in many works, e.g., [32, 46], where heat is defined as the change in the internal energy of the bath, i. e.

$$\Delta\tilde{Q} = \Delta E_B, \quad (\text{A1})$$

and no different types of energy are distinguished in this increase of energy. In this section, we compare these two definitions and argue why the approach taken here, though less extended, seems the most appropriate.

The ambiguity in defining heat comes from the different ways in which the change in the internal energy of the system E_S can be decomposed. More explicitly, let us consider a unitary process U_{SB} acting on a system-bath state ρ_{SB} with $\rho_B = \text{Tr}_S \rho_{SB} = \tau_B \propto e^{-H_B/kT}$ and global Hamiltonian $H = H_S \otimes \mathbb{I} + \mathbb{I} \otimes H_B$. The change in the total internal energy ΔE_{SB} is the sum of system and bath internal energies $\Delta E_{SB} = \Delta E_S + \Delta E_B$, or equivalently

$$\Delta E_S = \Delta E_{SB} - \Delta E_B. \quad (\text{A2})$$

Many text-books identify in this decomposition $\Delta E_{SB} := \Delta W$ as work and $\Delta E_B := \Delta\tilde{Q}$ as heat. Nevertheless, note that it also assigns to heat increases of the internal energy that are not irreversibly lost and can be recovered when having a bath at our disposal.

To highlight the incompleteness of the above definition, let us consider a reversible process $U_{SB} = \mathbb{I} \otimes U_B$ that acts trivially on the system. Then, even though the state of the system is untouched in such a process, the amount of heat dissipated is $\Delta\tilde{Q} = \Delta E_B = \text{Tr}[H_B(\rho_B - U_B \rho_B U_B^\dagger)]$.

In order to avoid this kind of paradoxes and in the spirit of the definition given above, we subtract from $\Delta\tilde{Q}$ its component of energy that can still be extracted (accessed). Then for a transformation $\rho_B \rightarrow \rho'_B$, the heat transferred is given as

$$\Delta Q = \Delta E_B - \Delta F_B, \quad (\text{A3})$$

$$= kT \Delta S_B, \quad (\text{A4})$$

where $\Delta F_B = F(\rho'_B) - F(\rho_B)$ is the work stored on the bath and can be extracted. Here, $F(\rho_X) = E_X - kT S(\rho_X)$

is the Helmholtz free energy, E_X is the internal energy and $\Delta S_B = S(\rho'_B) - S(\rho_B)$ is the change in the bath's von Neumann entropy, $S(\rho_B) = -\text{Tr}[\rho_B \log_2 \rho_B]$. Throughout this work, we consider \log_2 as the unit of entropy.

Let us finally remark that in practical situations, with large baths whose internal dynamics rapidly make them indistinguishable from thermal, both definitions coincide. However, when studying thermodynamics at the quantum regime with small machines approaching the nanoscale such conceptual differences are crucial to extend, for instance, the domain of standard thermodynamics to situations where the correlations become relevant.

Appendix B: Landauer's erasure principle (LEP): connecting heat and information

The information theory and statistical mechanics have long-standing and intricate relation. In particular, to exorcise Maxwell's demon in the context of statistical thermodynamics, Landauer first indicated that information is physical and any manipulation of that has thermodynamic cost. As put forward by Bennett [34], the Landauer information erasure principle implies that “[a]ny logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information-bearing degrees of freedom of the information-processing apparatus or its environment.”

Following the definition of heat, it indicates that, in such processes, *entropy increase in non-information-bearing degrees of freedom* of a bath is essentially associated with flow of heat to the bath. The major contribution of this work is to exclusively quantify heat in terms of flow of information, instead of counting it with the flow of non-extractable energy, the work. To establish this we start with the case of information erasure of a memory. Consider a physical process where an event, denoted with i , happens with the probability p_i . Then storing (classical) information memorizing the process means constructing a d -dimensional system (a memory-dit) in a state $\rho_S = \sum_i p_i |i\rangle\langle i|$, where $\{|i\rangle\}$ are the orthonormal basis correspond to the event i . In other words, memorizing the physical process is nothing but constructing a memory state $\rho_S = \sum_i p_i |i\rangle\langle i|$ from a memoryless state $|i\rangle\langle i|$ where i could assume any values $1 \leq i \leq d$. On the contrary, process of erasing requires the transformation of a memory state $\rho_S = \sum_i p_i |i\rangle\langle i|$ to a memoryless state $|i\rangle\langle i|$ for any i . Landauer's erasure principle (LEP) implies that erasing information, a process involving a global evolution of the memory-dit system and its environment, is inevitably associated with an increase in entropy in the environment.

In establishing the connection between information eras-

ing and heat dissipation, we make two assumptions to start with. First, the memory-system (S) and bath (B) are both described by the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_B$. Secondly, the erasing process involves entropy preserving operation Λ^{SB} , i.e., $\rho'_{SB} = \Lambda^{SB}(\rho_{SB})$. The latter assumption is most natural and important, as it preserves information content in the joint memory-environment system. Without loss of generality, one can further assume that the system and bath Hamiltonians remain unchanged throughout the erasing process, to ease the derivations.

Now we consider the simplest information erasing scenario, which leads to LEP in its traditional form. In this scenario, a system ρ_S is brought in contact with a bath ρ_B and the system is transformed to a information erased state, say $|0\rangle\langle 0|_S$, by performing a global entropy-preserving operation Λ^{SB} , i.e.,

$$\rho_S \otimes \rho_B \xrightarrow{\Lambda^{SB}} |0\rangle\langle 0|_S \otimes \rho'_B, \quad (\text{B1})$$

where initial and final joint system-bath states are uncorrelated. The joint operation guarantees that the decrease in system's entropy is exactly equal to the increase in bath entropy and heat dissipated to the bath is $\Delta Q = kT \Delta S_B$. It clearly indicates that an erasure process is expected to *heat up* the bath. This in turn also says that $\Delta Q = -kT \Delta S_S$, where $\Delta S_S = S(\rho'_S) - S(\rho_S)$. In the case where the d -dimensional system memorizes maximum information, or in other words it is maximally mixed and contains $\log_2 d$ bits of information, the process dissipates an amount $kT \log_2 d$ of heat to completely erase the information. In other words, to erase one bit of information system requires the dissipation of kT of heat and we denote it as one *heat-bit* or *ℓ -bit* (in honour of Landauer).

In the case where the final state may be correlated, the dissipated heat in general is lower bounded by the entropy reduction in the system, i.e.,

$$\Delta Q \geq -kT \Delta S_S. \quad (\text{B2})$$

This is what is generally known as the Landauer's erasure principle (LEP), in terms of heat.

The above formulation of LEP crucially relies on the fact that any change in system entropy leads to a larger change in the bath entropy, which is also traditionally known as the *second law* for the change in the information, i.e.,

$$\Delta S_B \geq -\Delta S_S. \quad (\text{B3})$$

However, it is limited by the assumptions made above and can be violated with initial correlations. Consider the examples proposed in the main text, see section IV. In both the examples, $\Delta S_B \not\geq -\Delta S_S$. Therefore, one has to replace it with universal informational second law (see Lemma 1).

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